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Confinement and the effective string theory in SU($N \to \infty$): a lattice study

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Abstract: We calculate in the SU(6) gauge theory the mass of the lightest flux loop that winds around a spatial torus, as a function of the torus size, taking care to achieve control of the main systematic errors. For comparison we perform a similar calculation in SU(4). We demonstrate approximate linear confinement and show that the leading correction is consistent with what one expects if the flux tube behaves like a simple bosonic string at long distances. We obtain similar but less accurate results for stable (k-)strings in higher representations. We find some evidence that for $k > 1$ the length scale at which the bosonic string correction becomes dominant increases as N increases. We perform all these calculations not just for long strings, up to about 2.5'fm' in length, but also for shorter strings, down to the minimum length, $l_c = 1/T_c$, where T_c is the deconfining temperature. We find that the mass of the ground-state string, at all length scales, is not very far from the simple Nambu-Goto string theory prediction, and that the fit improves as N increases from $N = 4$ to $N = 6$. We estimate the mass of the first excited string and find that it also follows the Nambu-Goto prediction, albeit more qualitatively. We comment upon the significance of these results for the string description of SU(N) gauge theories in the limit $N = \infty$.

Keywords: 1/N Expansion, Confinement, Lattice Gauge Field Theories.

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Contents

1. Introduction

There is considerable numerical evidence for linear confinement in SU(2) and SU(3) gauge theories and significant evidence that the effective string theory governing the long-distance dynamics of confining flux tubes is in the same universality class as the simplest Nambu-Goto bosonic string. (See [1]–[3] for some recent calculations.) For SU($N > 3$) gauge theories, while there is some evidence for linear confinement [1, 4], these questions have not been addressed with much precision. In particular, this is so for the stable k-strings in higher representations which have been the object of intensive recent study [1] and [4]–[6]. One usually assumes a simple bosonic string correction for these k-strings, and one also assumes that this leading correction dominates on the same length scales as it does for strings in the fundamental representation in SU(2) and SU(3). Since the comparison with theoretical predictions for k-string tensions is sensitive to this assumption, it is important to check its validity. In addition not much is known about the effective string theory at large N. Since it is in the $N = \infty$ limit that the gauge theory might be equivalent to a string theory [7] and since it is in that limit that one can currently make quantitative contact between field theories and their string theory duals [8], it is important to learn what are the properties of the effective confining string theory in that limit.

In this paper we address these questions with a lattice calculation in the SU(6) gauge theory. From other work on the mass spectrum and the deconfining temperature [9, 6, 10] we know that $N = 6$ is very close to $N = \infty$ for many quantities. This is no surprise since the leading large-N correction is expected to be $O(1/N^2)$. Our calculations are at a fixed

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value of the lattice spacing a. In units of the fundamental string tension σ, the value of a is $a \simeq 1/4\sqrt{\sigma}$ which is well into the weak coupling region where any corrections to continuum physics should be small. For comparison we also perform a calculation in the SU(4) gauge theory at a similar value of a.

The main questions we address are as follows.

- Do we have linear confinement at large N? To address this we calculate the energy of a 'flux tube' that winds around a spatial torus. We then vary the length of that torus and see if the energy grows (approximately) linearly with the length, once the length is large.
- What is the leading correction to this linear behaviour at large l? Equivalently: what is the central charge of the effective string theory that describes the long distance confining physics?
- If we reduce the length of the flux tube — indeed, to its minimum value — how does its energy compare to the behaviour predicted by the simple Nambu-Goto action? This addresses the question of what is the effective string theory on all scales. In particular, do things become simpler as $N \to \infty$?
- How does the energy of the first excitation of the string compare to the Nambu-Goto prediction? This probes the content of the effective string theory in a different way.
- How does all this affect the reliability of our calculations of k-string tensions? And indeed, what is our best estimate of these tensions?

In the next section we shall describe in greater detail the background to these questions. We then summarise the technical details of the lattice calculation. We then move on to the calculation itself. We begin by quantifying what are potentially the most serious systematic errors in such calculations. We then interpret our results. We finish with a summary of what we have learned and then describe the next step in improving upon these calculations.

2. Preliminaries

We shall begin with a discussion of how our calculations of the string energy as a function of length can probe the nature of the effective string theory. We then briefly outline the lattice setup of the calculations.

2.1 Strings

If we have linear confinement, we expect that the mass of a flux loop that winds around a spatial torus of length l will be given by [11, 12]

$$m(l) \stackrel{l\to\infty}{=} \sigma l - \frac{\pi}{3}\frac{c}{l}, \tag{2.1}$$

where c is the central charge of the effective string theory describing the long-distance properties of the confining flux tube. (The dimensional factor, $d-2$, has been included in the coefficient.) There is significant numerical evidence that $c = 1$ for SU(2) and SU(3) gauge theories [1]–[3], corresponding to the universality class of the simplest bosonic string. In this paper we shall attempt to determine the value of c for SU(6), in the expectation that this will also be its value at $N = \infty$.

There are old ideas that in its confining phase the SU($N = \infty$) gauge theory might be a string theory. It is therefore interesting to calculate $m(l)$ not just for large l but for all l, and indeed for excited strings as well. This is also interesting in the context of the more recent dual string approaches: while c in eq. (2.1) determines the number of exactly massless modes in the string theory, there will be massive modes generated in the descent to a boundary SU(∞) gauge theory, which may be encoded, for example, in the excited state spectrum of the string.

The very simplest bosonic string theory in d space-time dimensions, based on the Nambu-Goto action, has an energy spectrum [13, 14]

$$E_n(l) = \sigma l \left\{ 1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{d-2}{24}\right) \right\}^{1/2} \tag{2.2}$$

and a ground state energy, in d=4,

$$E_0(l) \stackrel{l\to\infty}{\longrightarrow} \sigma l - \frac{\pi}{3}\frac{1}{l} - \frac{\pi^2}{18\sigma}\frac{1}{l^3} + \cdots . \tag{2.3}$$

The correction to the universal $O(1/l)$ term in $E_0(l)$ is $O(1/l^3)$ and, as one would expect, the expansion is in inverse powers of $l^2\sigma$. This is consistent with the observation in [1] that the universal $O(1/l)$ term is the dominant correction for $l\sqrt{\sigma} \geq 3$. If one adds the most general higher-dimensional operators to the free term in the action, while remaining within an effective string theory with the same degrees of freedom, then one can show [14] that the expansion in eq. (2.3) retains its form except that the coefficient of the $O(1/l^3)$ term is now undetermined. If one includes in addition to interactions amongst the massless transverse modes, massive degrees of freedom which naturally arise in dual string constructions, then these would typically be encoded in excited states whose energies do not approach $E_0(l)$ as $l \to \infty$ [15]. Thus we can expect the string spectrum to contain interesting and useful information about the details of the equivalent string theory. In this spirit we will calculate the energy of the lightest and first excited strings as a function of l.

Of course, as one decreases l one encounters a minimum length for which a winding flux loop exists. This is easy to see. Suppose we choose the flux loop to wind around the x-torus. Now imagine relabelling co-ordinates $x, y, z, t \to t, x, y, z$. In this case the length l equals the inverse temperature and the mass of the winding string gives the effective string tension (times l) [16]. If T_c is the deconfining temperature then for $T > T_c$ there is no confinement and no winding string. Thus the minimum value of l is $l_c = 1/T_c$.

A linearly confining theory naturally embodies deconfinement because the number of strings of length l grows as $\propto \exp\{+\gamma l\}$ with $\gamma > 0$ and this will clearly overwhelm the Boltzman suppression $\propto \exp\{-\sigma l/T\}$ once T is large enough [17]. This argument naturally suggests a vanishing effective string tension at $T = T_c^s$ i.e. a second order transition. As we see from eq. (2.2), this occurs in the Nambu-Goto action at the length $l = 1/T_c^s$ where

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$E_0(l)$ vanishes, i.e. at

$$\frac{T_c^s}{\sqrt{\sigma}} = \sqrt{\frac{3}{(d-2)\pi}} \simeq \begin{cases} 0.691 : & d=4 \\ 0.977 : & d=3\,. \end{cases} \tag{2.4}$$

Of course this string deconfining temperature might be trumped by a transition to the gluon plasma at a lower value of $T = T_c < T_c^s$, driven by the large entropy of that plasma. In this case it is natural for the transition to be first order. It is therefore interesting to compare the predicted values in eq. (2.4) to the calculated values of T_c in those cases where the transition is in fact second order. One finds [16, 18]

$$\frac{T_c}{\sqrt{\sigma}} = \begin{cases} 0.709(4) : & \mathrm{SU}(2)\,, & d=4 \\ 1.12(1) : & \mathrm{SU}(2)\,, & d=3 \\ 0.98(2) : & \mathrm{SU}(3)\,, & d=3\,. \end{cases} \tag{2.5}$$

We note a quite remarkable co-incidence with the Nambu-Goto values. (This remark is not new: see e.g. [19].) Of course what we are really interested in is the $N \to \infty$ limit, where there are theoretical arguments that the gauge theory might be equivalent to a string theory, and here deconfinement is known to be first order [16]. Nonetheless this provides us with a bound

$$\lim_{N\to\infty} \frac{T_c^s}{\sqrt{\sigma}} > \lim_{N\to\infty} \frac{T_c}{\sqrt{\sigma}} = 0.596(4) \qquad d=4\,. \tag{2.6}$$

The Nambu-Goto string is of course extremely simple — no doubt much too simple. From string and dual string approaches [15] one expects

$$\frac{T_c^s}{\sqrt{\sigma}} \propto \frac{1}{\sqrt{c}}\,, \tag{2.7}$$

where c counts the number of effective degrees of freedom in the string theory, on the scale of T_c. Thus one typically would expect a value of $T_c^s/\sqrt{\sigma}$ that is lower than the Nambu-Goto value. And if one could obtain $\lim_{N\to\infty} T_c^s/\sqrt{\sigma}$ one might learn something interesting about the dual string theory. It might seem that this limit is inaccessible given that the transition is first order for all $N \geq 3$. However this is not necessarily so [20]. If the tension of the interface between confined and deconfined phases is $\propto N^2$ (its natural dependence) then the hysteresis around $T = T_c$ can become large [16, 21] on even moderately large spatial volumes. That is to say, if we are in the confined phase we will stay in that phase as we increase T above T_c and even as $T \to T_c^s$, if the hysteresis is extended enough. This is a realistic possibility. Preliminary calculations [20] in SU(8) indicate that for $N = 8$ the hysteresis is not (yet?) wide enough. Nonetheless the calculation provides an improved bound of $T_c^s/\sqrt{\sigma} > 0.63$.

One can extend the above discussion from strings in the fundamental representation to strings in higher representations [1]. It is useful to classify strings by how their sources transform under a gauge transformation belonging to the centre, $z \in Z_N$ for $\mathrm{SU}(N)$, since gluon screening does not transform sources between these classes. Let the transformation be z^k. Then the lightest string in this class is called the k-string. (The usual string is thus a 1-string.) One can think of it as a collection of k fundamental strings and an interesting question is whether they form a bound string state whose tension, σ_k, is less than the k separate strings, i.e. $\sigma_k < k\sigma$. It is also easy to see that k is less than the integer part of $N/2$ so that we have to go to at least SU(4) to have the possibility of this new type of string. In fact this possibility is realised [1]: the k-strings are strongly bound and the spectrum has been calculated quite accurately for $k \leq 4; N \leq 8$ [6]. Since the Nambu-Goto string action has no interactions, it cannot give such a binding and so cannot be the whole story at finite N. However as $N \to \infty$ the binding disappears, $\sigma_k \to k\sigma$, and the Nambu-Goto action re-emerges as a possibility. One might also entertain the possibility of the deconfining transition occurring in discrete steps at different temperatures, with the k-string deconfining at $T = T_c^k$, but it turns out that in practice this does not occur — there is a single first order transition for $N \geq 4$ [21]. And one can also argue that the same is the case for the would-be string deconfining temperatures $T_c^{s,k}$ [21].

The calculation of σ_k requires calculating the mass of a k-string of length l and then applying a correction as in eq. (2.1). Typically one assumes that this correction is the same as for the fundamental string, $c = 1$, and that it is the only important correction for $l\sqrt{\sigma} \geq 3$ because that is what one finds in SU(2) and SU(3) for the fundamental string. This assumption is important at the level of accuracy required to distinguish between competing theoretical ideas about the k-strings. Thus one of the things we will do is to try to obtain some direct numerical evidence with accurate calculations in SU(6) for the l-dependence of the masses of the lightest $k = 1, 2, 3$ strings.

2.2 Lattice

We work on periodic hypercubic $L_x \times L_y \times L_z \times L_t$ lattices with lattice spacing a. The degrees of freedom are SU(N) matrices, U_l, defined on the links l of the lattice. The partition function is

$$\mathcal{Z}(\beta) = \int \prod_l dU_l e^{-\beta \sum_p \{1 - \frac{1}{N} \mathrm{ReTr} u_p\}}; \qquad \beta = \frac{2N}{g^2} \tag{2.8}$$

where u_p is the ordered product of matrices around the boundary of the elementary square (plaquette) labelled by p and g^2 is the bare coupling. This is the standard Wilson plaquette action and since the theory is asymptotically free and since the bare coupling is a running coupling on length scale a, the continuum limit is approached by tuning $\beta = 2N/g^2(a) \to \infty$. One expects that for large N the value of a is fixed in physical units (e.g. in units of the mass gap) if one keeps the 't Hooft coupling $\lambda(a) \equiv g^2(a)N$ fixed i.e. $\beta \propto N^2$. This has been confirmed [9] in non-perturbative lattice calculations.

We will consider a loop of flux that winds around the x-torus, so that it is of length $l = aL_x$. (Our notation is that l denotes a length in either physical or lattice units, depending on the context, while the capitalized form is always in lattice units.) A generic operator ϕ_l that couples to such a periodic flux loop is an ordered product of link matrices along a space-like curve that winds once around the x-torus. Correlations are taken in the t direction so that the energy of the loop is an eigenstate of the hamiltonian (transfer

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matrix) defined on the xyz space. Such a correlator may be expanded

$$C(t = an_t) \equiv \langle \phi_l^\dagger(t)\phi_l(0)\rangle = \sum_{n=0} |\langle n|\phi_l(0)|vac\rangle|^2 e^{-aE_n n_t}\,; \qquad E_i \leq E_{i+1} \tag{2.9}$$

where, if we have confinement, E_0 is the energy of the lightest flux loop. Since the fluctuations that determine the error in the Monte Carlo calculation of $C(t)$ may be expressed as a correlation function with a disconnected piece, the error is approximately independent of t while $C(t)$ itself decreases exponentially with t. Thus one needs operators ϕ_l that have very large projections onto the desired state(s) so that this state dominates $C(t)$ at small t. This can be achieved using now standard techniques (see e.g. [6]). If we have linear confinement E_0 becomes large for long loops, and especially so for higher representations, and $C(t)$ becomes very small. It can then be difficult, even at small t, to make the error on $C(t)$ much smaller than its value, if one uses a standard Monte Carlo. In some of these cases we therefore use a recently developed error reduction technique [22].

We choose $L_y = L_z$ for convenience. We need to make sure that L_y is large enough for there to be no significant finite volume effects. We will also need to choose L_t large enough so that the only state with a significant probability to propagate around the t-torus is the vacuum state. In practice it turns out that for large L_x one can use $L_y = L_z = L_t = L_x$, but for small L_x we shall need to use $L_y, L_t \gg L_x$. Controlling these different finite volume effects is one way in which the present calculation improves upon previous calculations.

Our calculations will be performed for a single lattice spacing. In the case of SU(6) we choose $\beta = 25.05$. This corresponds to

$$a(\beta = 25.05) = \frac{0.252(1)}{\sqrt{\sigma}} = \frac{0.1509(5)}{T_c} \quad : \qquad \mathrm{SU}(6) \tag{2.10}$$

in units of the fundamental string tension σ (see below) or the deconfining temperature T_c [16]. In SU(4) we choose $\beta = 10.90$, which corresponds to

$$a(\beta = 10.9) = \frac{0.241(2)}{\sqrt{\sigma}} = \frac{0.1502(6)}{T_c} \quad : \qquad \mathrm{SU}(4)\,. \tag{2.11}$$

The value of the lattice spacing has been chosen to be (almost) the same for SU(4) and SU(6) so that we can separate the N-dependence from the a-dependence (to leading order). This value of a is small enough that we can expect lattice corrections to be negligible for our purposes.

3. Results

We calculate the string mass as a function of its length on a variety of lattice volumes. In table 1 we list the masses obtained in SU(6) at $\beta = 25.05$ for the lightest $k = 1, 2, 3$ strings and, on the largest lattices, for the first excited $k = 1$ string. In table 3 we list a less extensive set of results obtained in SU(4) at $\beta = 10.9$.

We will begin by describing the important sources of systematic error in such calculations and we show how the results listed in Table 1 enable us to control these errors. Having obtained the string energies $E_0^k(l)$ for l ranging from close to its minimum possible value upwards, we address the questions posed in the Introduction.

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SU(6)						
l_x	$l_{y,z}$	l_t	$am(k=1)$	$am^\star(k=1)$	$am(k=2)$	$am(k=3)$
7	12	36	0.19(3)		0.24(11)	0.22(16)
7	16	40	0.242(6)	1.635(66)	0.488(16)	0.624(23)
8	8	14	0.295(6)		0.414(17)	0.455(24)
8	8	30	0.302(8)		0.480(19)	0.534(28)
8	10	30	0.343(8)	1.244(31)	0.582(19)	0.703(41)
8	12	30	0.3343(50)	1.234(17)	0.596(13)	0.710(18)
10	10	10	0.494(6)		0.800(17)	0.973(31)
10	10	16	0.505(9)	1.392(21)	0.863(19)	1.039(27)
10	12	16	0.5141(50)	1.366(19)	0.907(10)	1.052(18)
12	12	12	0.669(6)	1.497(25)	1.148(16)	1.354(29)
12	14	14	0.6683(55)	1.505(24)	1.180(13)	1.422(23)
16	16	16	0.9465(59)	1.638(22)	1.625(20)	1.878(45)
$16^\star$	16	24	0.930(10)	1.627(17)	1.634(19)	1.844(60)
20	20	24	1.210(7)			2.24(8)
$20^\star$	20	24	1.242(15)	1.800(51)	2.077(67)	

Table 1: Masses of closed strings of length l_x in SU(6).

SU(6) : $\sigma l - c\pi/3l$ fits						
$l_x \geq$	$c(k=1)$	χ^2/n_{df}	$c(k=2)$	χ^2/n_{df}	$c(k=3)$	χ^2/n_{df}
7	1.36(4)	3.8	2.02(12)	2.1	1.96(19)	5.4
8	1.31(5)	4.0	2.11(15)	2.3	2.26(25)	5.7
10	1.16(8)	0.9	1.73(22)	0.4	1.48(42)	5.7
12	1.09(10)	0.7	1.45(40)	0.0	-0.38(73)	0.8

Table 2: Fits to SU(6) string masses.

SU(4)					
l_x	$l_{y,z}$	l_t	$am(k=1)$	$am^\star(k=1)$	$am(k=2)$
7	20	60	0.1655(75)	1.50(5)	0.261(16)
8	16	24	0.3015(50)		0.470(15)
10	16	24	0.4506(69)		0.698(13)
12	16	24	0.5928(88)	1.375(19)	0.856(15)
16	16	24	0.8610(53)	1.571(8)	1.2102(63)

Table 3: Masses of closed strings of length l_x in SU(4).

3.1 Systematic errors

There are a large number of potential systematic errors. Here we focus on three that we believe are potentially the most important ones in this kind of calculation.

3.1.1 Excited state contamination

Our variational calculation produces operators $\Phi_l^k(t)$ that have very good but not perfect overlaps onto the ground string states. We therefore calculate the correlator $C(t) = \langle \Phi_l^{k\dagger}(t)\Phi_l^k(0)\rangle$ and extract the energy from the larger values of $t = an_t$ where the excited state contamination has died out. The signal that we have reached such t is that the correlator is given by a single exponential. A problem arises if $E_0^k(l)$ is large — either because l is large or because k is large. In this case the correlator will be very small beyond the smallest values of t, so that it becomes dominated by the statistical error and one cannot obtain any significant evidence that the higher excited states have in fact died away at the t at which one extracts the energy. Clearly the bias will be to extract energies at lower t as the energy increases and so to risk a larger admixture of even heavier excited states, and so to overestimate the energy as k or l increases. This risk is enhanced for larger k because the best overlap decreases roughly as the k'th power of the $k = 1$ overlap. (Which provides a motivation for our description of the k-string as a loosely bound state of k fundamental strings.) It is also enhanced for large l by two facts. First the gap between ground and excited states decreases as $\propto 1/l$ and second the overlap onto the lightest string appears to decrease exponentially with its length l. The second effect is not unexpected and is in practice not severe: the overlap reduces from ~ 0.985 for $l = 8$ to ~ 0.960 for $l = 16$ using a very similar basis of operators.

All this is not an issue for the smallest values of l where we have accurate calculations of $C(t)$ over a large range of t. To obtain some control of this error for larger l, in particular for $l = 16$ and $l = 20$, we have performed not only a normal Monte Carlo calculation of the correlator, but also a separate calculation using a novel and powerful error reduction technique [22, 23]. The latter calculations are starred in table 1. The error reduction technique works best for a particular mass range for which it needs to be tuned in advance. Here we have tuned the calculation to work best for masses close to those of the lightest $k = 2$ string. The unstarred calculations use a standard Monte Carlo algorithm and, although the masses are extracted from an exponential fit to $C(t)$ between $t = a$ and $t = 4a$, in the case of $l = 16$ and $l = 20$ the signal-to-error ratio grows so rapidly with increasing t that, effectively, the masses are obtained from $t = a$ to $t = 2a$. The starred calculations involve extracting the masses from $t = 2a$ to $t = 3a$ where we can be quite confident that any contamination from heavier excited string states will be negligible.

Comparing the results from the starred and unstarred calculations in table 1 we find very good agreement within the pairs of calculations for all the states. This provides good direct evidence that any contamination from excited states is negligible in the present calculation.

3.1.2 Transverse size corrections

Calculations of winding flux loops have usually been performed on L^3L_t lattices. Is a transverse size of L large enough for a flux tube of length L or does this create significant finite volume corrections? To the extent that L is large compared to the width of the flux tube, and the oscillations are simple single-valued harmonic modes, one would expect the corrections to be very small. So unlike the systematic error discussed above, this is expected to affect short rather than long strings. And to the extent that a k-string is like k weakly bound $k = 1$ strings, we might expect any error to be more severe for larger k.

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To address this question we start by comparing the three calculations in table 1 that have $L_x = 8$ and $L_t = 30$. (The corrections induced by too small a value L_t will be discussed below.) These three calculations have transverse sizes $L_y = L_x = 8$, $L_y = 10$ and $L_y = 12$. We observe that there is no discrepancy in the masses obtained with the last two, implying that for $L_x = 8$ a transverse size of $L_y = 10$ is in fact large enough. The $L_y = L_x = 8$ calculation, on the other hand, produces masses that are very different: while for $k = 1$ the shift is a modest $\simeq 10\%$, for $k = 2, 3$ the shifts are $\sim 24\%$ and $\sim 35\%$ respectively. Our expectation that the corrections should be larger for larger k thus appears to be confirmed.

Moving to $L = 10$ we have $L_y = 10$ and $L_y = 12$ to compare. For $k = 1$ there appears to be no difference, while there is evidence for a modest discrepancy $\sim 5\%$ for the $k = 2$ string. That this is not merely a statistical fluctuation is confirmed if we compare the $k = 2$ effective masses obtained from the correlator between $t = 0$ and $t = a$, where the relative statistical error is much smaller. The conclusion is that for $L = 10$ there is a shift in using $L_y = 10$ but that it is very small. Moving on to $L = 12$ we compare $L_y = 12$ and $L_y = 14$. The lightest and first excited $k = 1$ strings show no difference and the possible difference for the $k = 2, 3$ strings is shown to be no more than a statistical fluctuation when we look at the statistically much more accurate $t = 0, a$ effective masses.

We therefore conclude that, for this value of a, once $L \geq 12$ it is safe to use an $L^3 L_t$ lattice for such string calculations. Using $a\sqrt{\sigma} \simeq 0.25$ this translates into the statement that if the length $l = aL_x$ satisfies $l\sqrt{\sigma} \geq 3$ then an equal transverse size is adequate. It so happens that, for quite different reasons, lattices used for calculating k-string tensions have satisfied precisely this criterion.

3.1.3 Temporal size corrections

If all the spatial dimensions, L_i, are large then all the string energies will be large, and $\exp\{-EL_t\}$ will be negligibly small for all states other than the vacuum if we choose, for example, $L_t = L_i$. (Recall that all glueball masses will in practice be large compared to $1/L_t$.) So for larger L_i the partition function is indeed dominated by the vacuum and there is no problem. For small L, however, we can only be sure of excluding the spatially periodic strings from propagating right around the temporal torus if we choose L_t so that $\exp\{-E_0(L)L_t\} \ll 1$. This means that we must increase L_t as we decrease L. If we do not do so, then the partition function, which is a normalising factor within all our calculated expectation values, will no longer dominated by the vacuum contibution and we run the risk that our various calculated energies will no longer be with respect to the correct vacuum energy.

In practice one finds that for the $k = 1$ string such an effect is quite weak — as we see by comparing the $8^3 14$ and $8^3 30$ lattices in table 1 or the 10^4 with the $10^3 16$. This is as expected: e.g. for the $8^3 14$ lattice one sees that $\exp\{-E_0(L)L_t\} \sim 0.015$ and there is a factor of 3 for the strings in the different spatial directions. So the expected shift in the effective vacuum energy is only $\Delta E_{vac} \sim 0.003$. In fact it is well known that even with smaller L_t the corrections are smaller than expected, and this can be ascribed to a cancellation between these extra modes in the path integral numerator and in normalising denominator

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(partition function) in the calculation of an expectation value. One would expect such a cancellation if the interaction between this extra mode and the string propagating between the operators of the correlator was weak.

However, as we see in table 1, there is a quite dramatic mass shift for the $k = 2$ and $k = 3$ masses when we compare, for example, the $8^3 14$ and $8^3 30$ lattices. We can interpret this, in part, as follows. Consider our $k = 2$ calculation. We have two $k = 2$ operators separated by t and we assume that for large enough t the correlator is dominated by the lightest $k = 2$ loop propagating between these operators. However an alternative contribution arises from two $k = 1$ loops propagating between the operators in opposite directions around the temporal torus. Such a contribution is t-independent and acts, in the calculation, like an unexpected non-zero vacuum expectation value for the $k = 2$ loop operator. This will lower the calculated effective mass at larger t. It is therefore clear from this that we require L_t to be large enough that $\exp\{-E_0^{k=1}(L)L_t\} \ll \exp\{-E_0^k(L)t\}$ for any value of t at which we might evaluate the mass of the k-string.

3.2 Interpretation

Our above discussion of systematic errors suggests that the following choice of lattices from table 1 will provide reliable string masses. For $l = 7, 8, 10$ we use the $7 \times 16^2 \times 40$, $8 \times 12^2 \times 30$ and , $10 \times 12^2 \times 16$ lattices respectively. For $l \geq 12$ we average over the pairs of calculations at each l.

In figure 1 we plot the lightest and first excited string masses versus their length l. The first thing we observe is that the lightest loop mass, $E_0(l)$, grows more-or-less linearly with its length, l. But before claiming that this provides direct evidence for linear confinement we need to establish that this is not some sub-asymptotic behaviour at short distances. Now, if we extract a string tension (see below) we obtain $a\sqrt{\sigma} \simeq 0.25$. Thus our largest flux loop length is $20a \simeq 5/\sqrt{\sigma} \simeq 2.5$'fm'. (The 'fermi' units are obtained using the QCD value of the string tension and are only introduced to provide an intuitively familiar scale of length.) This is long enough to make it plausible that we are indeed seeing the asymptotic linear rise of a linearly confining SU(6) gauge theory. Since we expect corrections to the SU(∞) theory to be $O(1/N^2)$ and since the extracted value of the string tension is very similar to that in, for example, SU(3) when calculated in units of, for example, the mass gap [9, 6], this provides very plausible evidence for linear confinement in the $N = \infty$ gauge theory.

We now attempt to fit the flux loop mass with a leading linear term and a $O(1/l)$ string correction as in eq. (2.1). Since these are supposed to be the leading terms at large l, it is not surprising that we do not obtain an acceptable fit over the whole range of l. We therefore drop lower l values from the fit and find that we first obtain an acceptable χ^2 per degree of freedom, χ^2_{df}, for $l \geq 10$, with a slightly better fit for $l \geq 12$:

$$aE_0(l) = a^2\sigma l - \frac{c\pi}{3l} = \begin{cases} 0.06383(38)l - 1.16(8)\pi/3l : & l \geq 10\,, \quad \chi^2_{df} = 0.85 \\ 0.06356(47)l - 1.09(10)\pi/3l : & l \geq 12\,, \quad \chi^2_{df} = 0.70\,. \end{cases} \tag{3.1}$$

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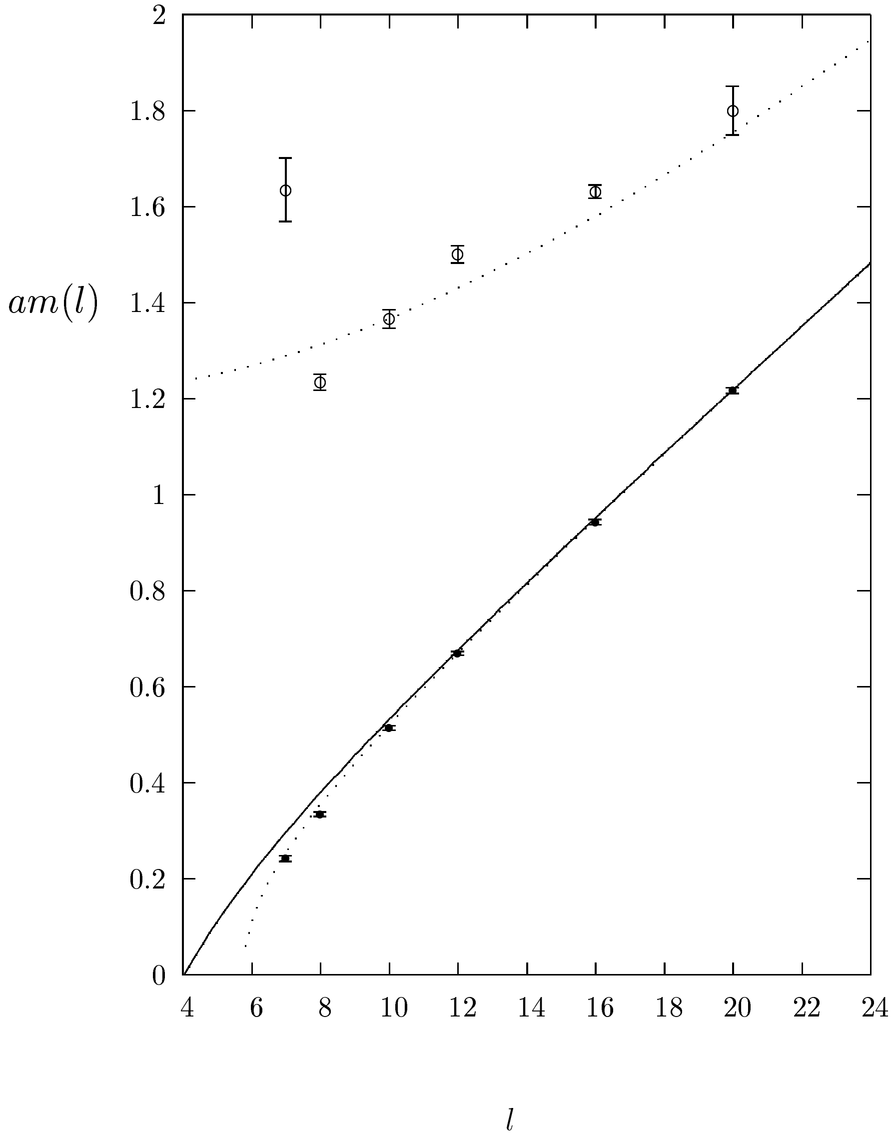

Figure 1: The masses of the lightest, $\bullet$, and first excited, $\circ$, $k = 1$ flux loops that wind around a spatial torus of length l in the SU(6) calculation at $\beta = 25.05$. The dotted lines are the predictions of the Nambu-Goto string action, as in eq. (2.2). The dynamical lower bound on the string length is $l_{\min} = 1/aT_c \simeq 6.63$.

Defining an effective value, c_{eff}, of the string coefficient obtained between neighbouring values of l

$$c_{\text{eff}} \frac{\pi}{3} \left\{ \frac{1}{l_1^2} - \frac{1}{l_2^2} \right\} = \frac{aE_0(l_2)}{l_2} - \frac{aE_0(l_1)}{l_1}, \tag{3.2}$$

we show in figure 2 how it varies with increasing l. All this provides significant evidence that the effective string theory for long flux tubes is in the universality class of a simple bosonic string theory, $c = 1$.

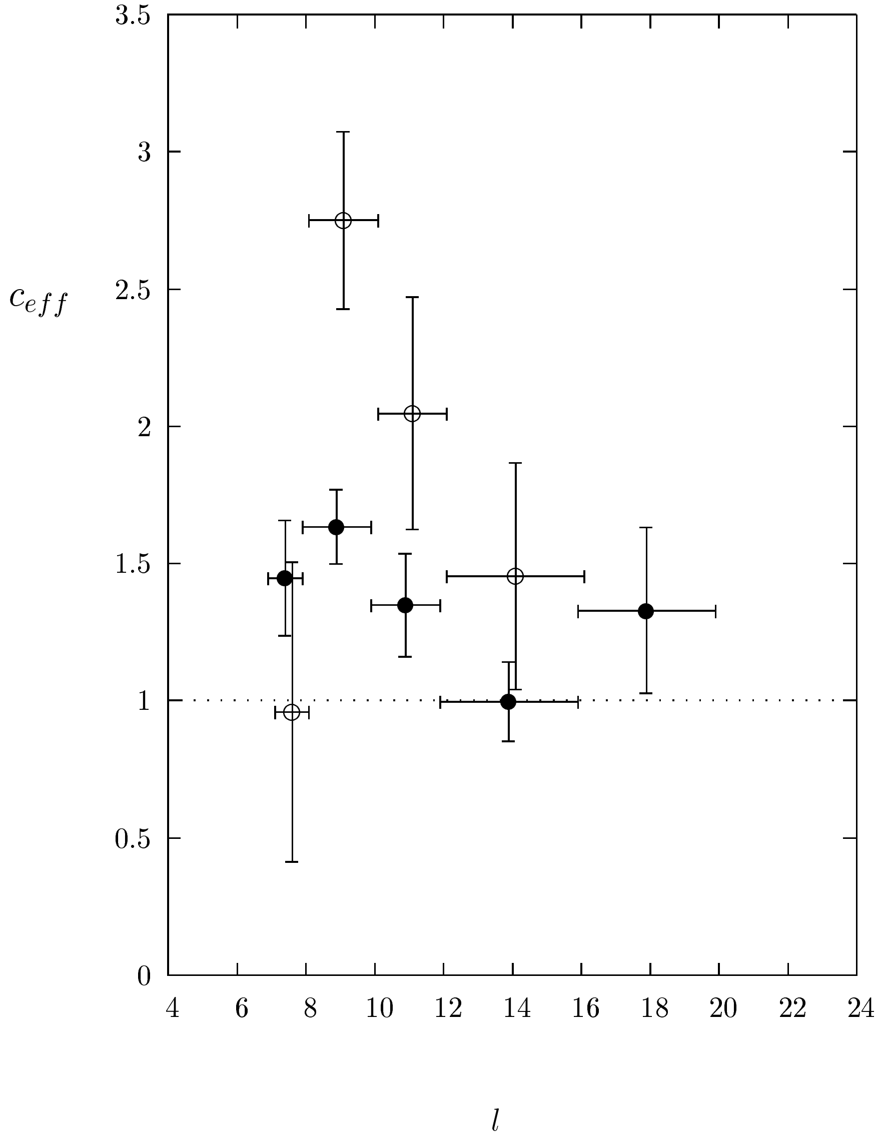

Figure 2: The effective coefficient of the $1/l$ universal string correction, calculated from eq. (3.2), for the range of lengths indicated. For $k = 1$, $\bullet$, and $k = 2$, $\circ$, strings in SU(6).

There will of course be higher order corrections in $1/l$ and, under certain natural assumptions, the next correction will be down by $1/l^2$ [14]. If we fix the coefficient of the $1/l$ term to the bosonic string value $\pi/3$, then we find that we can obtain an acceptable fit to our whole range of l:

$$aE_0(l) = 0.06358(20)l - \frac{\pi}{3l} - \frac{19.4(3.2)}{l^3} \; : \quad l \geq 7\,, \quad \chi^2_{df} = 0.9\,. \tag{3.3}$$

This is remarkable since, as pointed out earlier, there is a minimum length for a periodic flux loop, which is $l_{\min} = 1/aT_c \simeq 6.66$ in the present calculation. Thus the fit in eq. (3.3) essentially works all the way down to the shortest possible strings. (If we fit the $O(1/l)$ term as well, then its coefficient comes to 0.94(16), with a slightly worse value of χ^2_{df}.) Note that the natural dimensionless expansion parameter is $1/\sigma l^2$ and so we would expect the $1/l^3$ term in eq. (3.3) to have a coefficient that is larger by a factor $O(1/\sigma) \sim 16$ than that of the leading $1/l$ correction. This is indeed consistent with what we find. Note also that the $O(1/l^3)$ coefficient is about twice as large as the Nambu-Goto coefficient in eq. (2.2).

In figure 1 we also plot the prediction of the Nambu-Goto string theory for $E_0(l)$, as in eq. (2.3) with $n = 0$. This fit has only one free parameter, σ, and visually follows the calculated values quite closely although, with $\chi^2_{df} \simeq 3.0$, it is clearly not an acceptable fit. However, for the comparable SU(4) calculation (discussed below) we find $\chi^2_{df} \simeq 7.5$. This improvement in goodness of fit with increasing N does leave open the intriguing possibility that at $N = \infty$ the effective string theory might be governed by the simple Nambu-Goto action.

We also plot the prediction from eq. (2.2) for the first excited state, which now has no free parameter at all. We see that this tracks the calculated values quite well, except at the very smallest l. It is interesting to note that the gap between the energy of the lightest and first excited string state is large, and for most values of l larger than the mass of the lightest glueball, which has a mass $am_G \simeq 0.74$. For such l the excited string can presumably decay into the lightest string and a glueball. This should show up in the correlator through an effective energy that gradually decreases with increasing t. The fact that we see no sign of this is presumably due to the large-N suppression of decays. For SU(4), where decays are not so suppressed, we do in fact find it difficult to extract excited string energies for intermediate l, and it is possible that this is part of the reason.

In figure 3 we plot the masses of the lightest $k = 2$ and $k = 3$ strings. These are heavier than the $k = 1$ string and thus possess larger errors. Nonetheless the approximate linear growth with l is evident. We plot fits with the bosonic string correction and we list the best coefficients of a $1/l$ correction in table 2. The $k = 3$ calculation is not accurate enough to be informative on this issue, but for the $k = 2$ string we have some statistically weak evidence that for $l \geq 12a \simeq 3/\sqrt{\sigma}$ the dominant correction is $O(1/l)$, with a coefficient that is consistent with that of the simple bosonic string. To investigate this in more detail we analyse the $k = 2$ string masses using eq. (3.2), and plot the result in figure 2. Comparing the values of c_{eff} from the $k = 1$ and $k = 2$ strings, we see that the $k = 2$ values are much larger for shorter strings, and, while consistent with approaching the bosonic string value, probably does so for somewhat larger values of l. This behaviour is, in fact, not unexpected if we think of the $k = 2$ string as two loosely bound $k = 1$ strings. If it acts as two incoherent, unbound strings it receives twice the string correction that it would receive if it behaved as a coherent, single string. Since the correction is negative and grows with decreasing l, it is clear that there should be some length l' such that for $l < l'$ it will be energetically favourable for the k-string to be k incoherent $k = 1$ strings rather than a single coherent k-string. Moreover since $\sigma_k \to k\sigma$ as $N \to \infty$, it is also clear that $l' \to \infty$ as $N \to \infty$ [6]. More generally we can think of the Lüscher correction as inducing a repulsive interaction between the two strings, which leads to a value of c_{eff} that smoothly decreases from 2 to 1 at a length scale that grows with N. It is tempting to see figure 2 as displaying such a behaviour.

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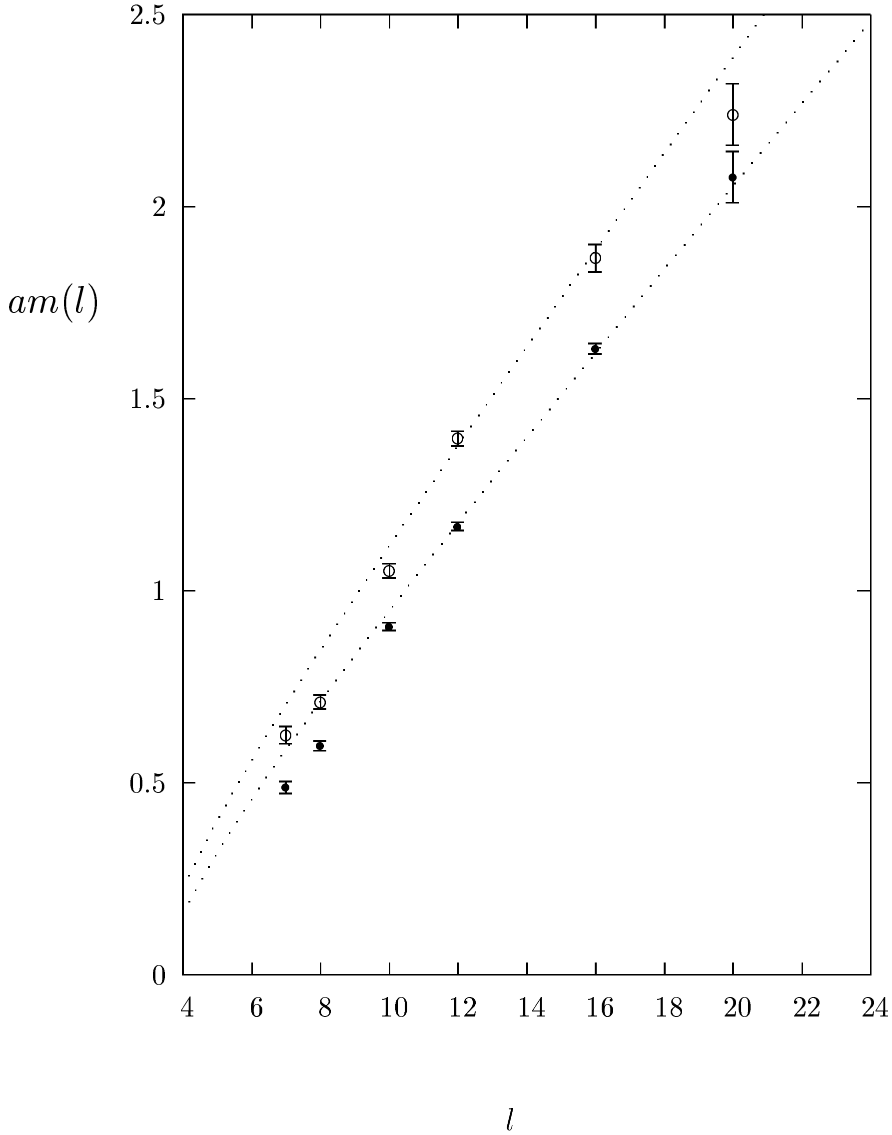

Figure 3: The masses of the lightest $k = 2$, $\bullet$, and $k = 3$, $\circ$, flux loops that wind around a spatial torus of length l in the SU(6) calculation at $\beta = 25.05$. The dotted lines are the best fits with a bosonic string correction, as in eq. (2.1) with $c = 1$.

In addition to the above SU(6) calculation we have also performed, for comparison, a less extensive and less accurate SU(4) calculation, as listed in table 3. A fit with a $O(1/l)$ bosonic string correction plus an extra $O(1/l^3)$ correction gives

$$aE_0(l) = 0.05798(32)l - \frac{\pi}{3l} - \frac{17.2(4.8)}{l^3} \;:\quad l \geq 8\,, \quad \chi^2_{df} = 1.0\,. \tag{3.4}$$

As with SU(6) the coefficient of the $O(1/l^3)$ is of the expected order of magnitude, although it is no longer possible to include the smallest value of l in the fit. In figure 4 we plot the lightest and first excited $k = 1$ flux loop masses and compare to the Nambu-Goto string predictions. As we remarked earlier, the best Nambu-Goto fit is considerably worse than for SU(6). It is thus possible that such a fit will become acceptable in the $N = \infty$ limit.

Fitting the $k = 2$ string masses with an $O(1/l)$ bosonic string correction and an additional $O(1/l^3)$ term, we obtain as acceptable fits,

$$aE_0^{k=2}(l) = \begin{cases} 0.07963(53)l - \pi/3l + 1.7(21.2)/l^3 : & l \geq 10\,, \quad \mathrm{SU}(4) \\ 0.10739(75)l - \pi/3l - 65.7(10.6)/l^3 : & l \geq 8 \quad \mathrm{SU}(6)\,. \end{cases} \tag{3.5}$$

The fact that the $O(1/l^3)$ coefficient is much larger for $N = 6$ than for $N = 4$ presumably encodes the fact that the minimum string length at which the Lüscher correction becomes a good approximation is larger for larger N. (And the fact that it is much larger than for $k = 1$ in eq. (3.3) suggests it grows with k.) Together with eqs. (3.3), (3.4), these lead to

$$\frac{\sigma_k}{\sigma} \stackrel{a\simeq 0.15/T_c}{=} \begin{cases} 1.373(12) : & \mathrm{SU}(4) \\ 1.689(13) : & \mathrm{SU}(6) \end{cases} \tag{3.6}$$

These ratios have very small systematic and statistical errors and show quite clearly that at this fixed but small value of the lattice spacing these string tensions lie in between the Casimir scaling [24] and 'MQCD' conjectures [25] with which they are usually compared [1, 5, 6].

As an aside, we note that the above calculations also tell us about how the effective string tension, σ_{eff}, varies with temperature T. Let us relabel our euclidean axes so that our x direction is the t direction. Then our $k = 1$ Wilson line operator is just the world line of a static fundamental source and if we define σ_{eff} to be the coefficient of the linear piece of the free energy of two such static sources then it is clear that [16, 21]

$$\sigma_{\mathrm{eff}}(T) \stackrel{l=1/T}{=} \frac{E_0(l)}{l} \stackrel{l=1/T}{=} \sigma - \frac{\pi}{3}T^2 + O(T^4) \tag{3.7}$$

if the string is in the simplest bosonic universality class. In figure 5 we plot the values of $\sigma_{\mathrm{eff}}(T)$ against T/T_c for both SU(4) and SU(6). There is a hint that as N increases the effective string tension has smaller corrections at all T.

4. Conclusions

In our calculation in the SU(6) gauge theory we have obtained good evidence for linear confinement of flux loops that are up to $l \simeq 5/\sqrt{\sigma}$ long. This is long enough in physical units (it corresponds to $\sim 2.5fm$ in QCD) for us to see it as good evidence for exact linear confinement. The calculation is at a fixed lattice spacing but this is small enough, $a \sim 1/4\sqrt{\sigma}$, that we are confident that we are seeing continuum physics. Finally $N = 6$ is large enough that we see this is as good evidence for linear confinement in the $N \to \infty$ limit.

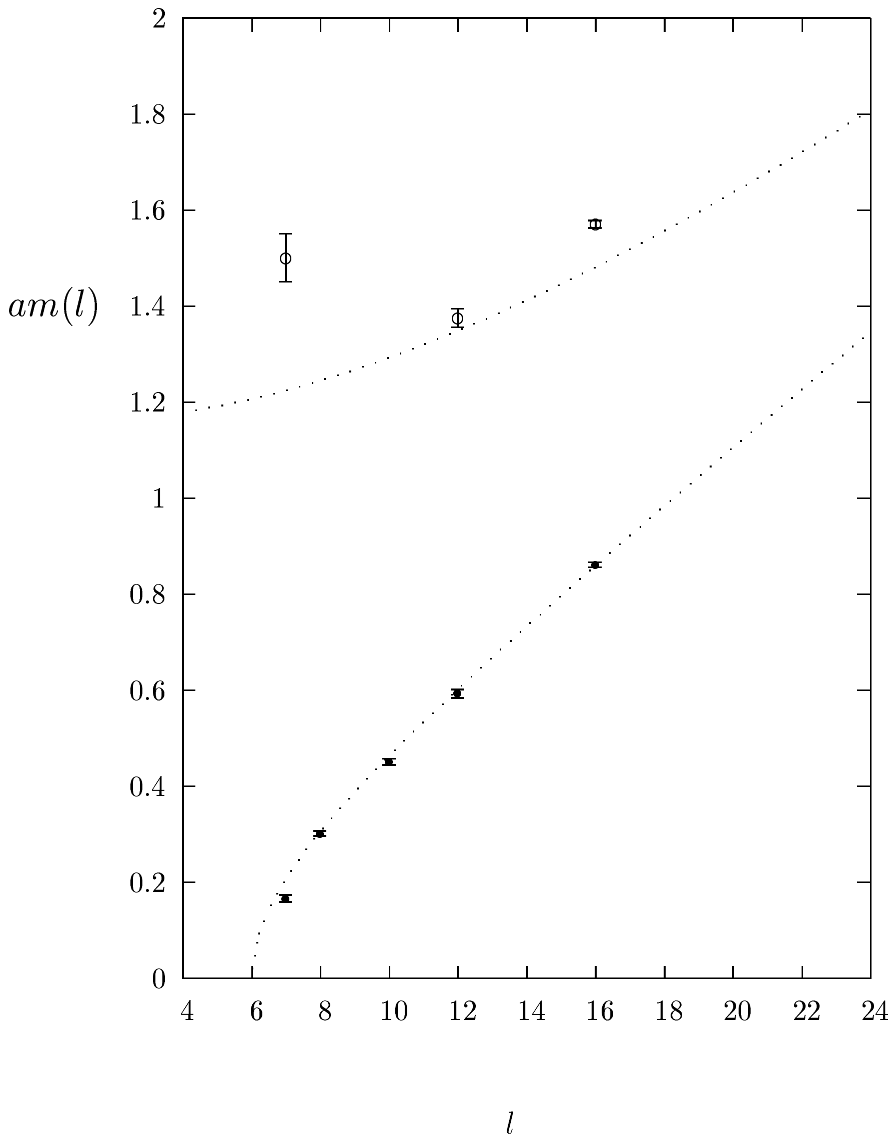

Figure 4: The masses of the lightest, •, and first excited, ∘, $k = 1$ flux loops that wind around a spatial torus of length l in the SU(4) calculation at $\beta = 10.9$. The solid line is the best fit with a bosonic string correction, as in eq. (2.1) with $c = 1$. The dotted lines are the predictions of the Nambu-Goto string action, as in eq. (2.2). The dynamical lower bound on the string length is $l_{\min} = 1/aT_c \simeq 6.66$.

For strings in the fundamental representation our SU(6) calculation shows that the leading $O(1/l)$ correction at large l has the coefficient one expects if the effective string theory belongs to the simplest bosonic string universality class. This coefficient is determined accurately enough for this result to be significant.

These calculations were performed with care to control and quantify the main systematic errors. We found that string tensions could be reliably calculated on L^4 lattices once $L \geq 3/a\sqrt{\sigma}$. This fortunately coincides with the conventional choice of lattice size for such

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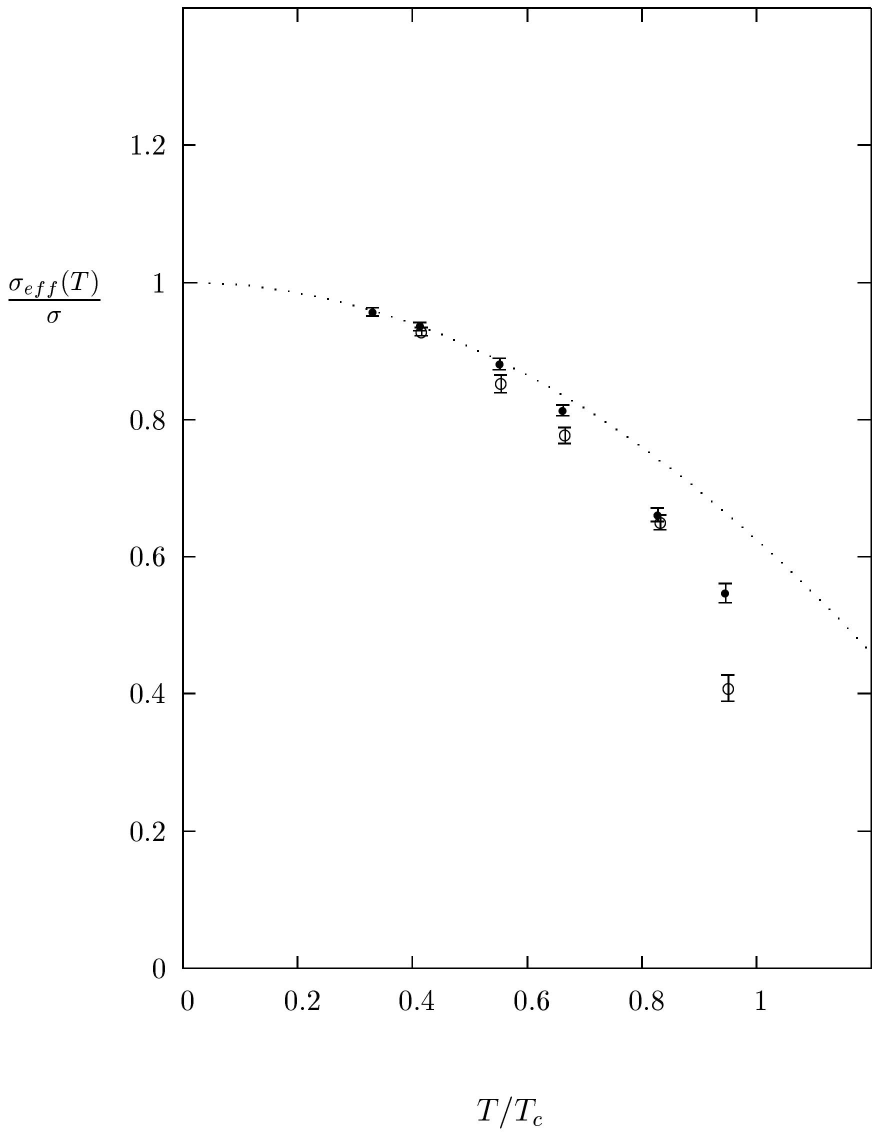

Figure 5: The finite temperature effective string tension $\sigma_{\text{eff}}(T)$ plotted as a function of T/T_c for SU(6), $\bullet$, and SU(4), $\circ$, at a fixed lattice spacing $a \simeq 0.15/T_c$.

calculations [1]. For flux loops in higher representations our results do not pin down the string corrections very accurately, but they are consistent with a bosonic string correction dominating at large l. There is, however, evidence that this occurs at larger l than for the $k = 1$ string, the more so the larger N, and that the effective string coefficient at smaller l is larger for larger k. This fits in with the picture of a k string as being k loosely bound fundamental strings with the Lüscher correction inducing an effective repulsive interaction. It also implies that past calculations of k-string tensions may have suffered from a small but systematic downward bias. The ratio of the $k = 2$ and $k = 1$ string tensions that we obtain in this paper lies more-or-less mid-way between the Casimir Scaling and the 'MQCD' motivated trigonometric formula.

Remarkably, we found that using only the $O(1/l)$ bosonic string correction with an additional $O(1/l^3)$ term, we could describe the (fundamental) string energy all the way down to the minimum string length, $l \simeq 1/T_c$ where T_c is the deconfining temperature. The fitted coefficient of the $O(1/l^3)$ term turns out to be $O(1/a^2\sigma)$ just as one would expect on general grounds.

The string energy is also found to be close to the prediction of the simplest string theory, that governed by the Nambu-Goto action, and the comparison between SU(4) and SU(6) leaves open the possibility that for SU(∞) the effective string theory is indeed governed by that action — although this would be hard to understand on theoretical grounds [7, 8]. Our calculation of the energy of the first excited string and our discussion of string deconfinement, reinforces the conclusion that whatever effective string action governs the $N = \infty$ theory, it must be 'quite close' to Nambu-Goto.

It would be useful to have, say, SU(4) and SU(8) calculations that are as precise as our SU(6) ones, so that we could directly extrapolate the string energy to $N = \infty$ as a function of l, all the way down down to the minimum length, $l \simeq 1/T_c$, dictated by 'spatial deconfinement'. In fact, as we remarked earlier, one can use the hysteresis of this first order phase transition to go to lower l. The aim would be to go all the way down to the second order string condensation transition where the string mass vanishes. Although it appears that the hysteresis is not wide enough for this to be possible in SU(8), there is good reason to believe it will be possible for not very much larger values of N. The low-l behaviour of the string energy together with the excited string spectrum, particularly for states whose energies do not approach the ground state energy at large l, should provide useful insights into the detailed nature of the effective string theory for, and the dual string theory to, the SU(∞) gauge theory.

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References

[1] B. Lucini and M. Teper, *Confining strings in* SU(N) *gauge theories*, *Phys. Rev.* **D 64** (2001) 105019 [hep-lat/0107007].

[2] M. Lüscher and P. Weisz, *Quark confinement and the bosonic string*, *J. High Energy Phys.* **07** (2002) 049 [hep-lat/0207003].

[3] K.J. Juge, J. Kuti and C. Morningstar, *QCD string formation and the Casimir energy*, hep-lat/0401032;
P. Majumdar, *Continuum limit of the spectrum of the hadronic string*, hep-lat/0406037.

[4] B. Lucini and M. Teper, *The $k = 2$ string tension in four dimensional* SU(N) *gauge theories*, *Phys. Lett.* **B 501** (2001) 128 [hep-lat/0012025].

[5] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, *k-string tensions in* SU(N) *gauge theories*, *Phys. Rev.* **D 65** (2002) 021501 [hep-th/0106185]; *Spectrum of confining strings in* SU(N) *gauge theories*, *J. High Energy Phys.* **01** (2002) 009 [hep-th/0111090].

[6] B. Lucini, M. Teper and U. Wenger, *Glueballs and k-strings in* SU(N) *gauge theories: calculations with improved operators*, *J. High Energy Phys.* **06** (2004) 012 [hep-lat/0404008].

[7] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461; *A two-dimensional model for mesons*, *Nucl. Phys.* **B 75** (1974) 461;
A.M. Polyakov, *Gauge fields and strings*, Harwood, 1987;
J. Polchinski, *Strings and QCD?*, hep-th/9210045.

[8] J.M. Maldacena, *TASI 2003 lectures on AdS/CFT*, hep-th/0309246;
O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[9] B. Lucini and M. Teper, SU(N) *gauge theories in four dimensions: exploring the approach to* $N = \infty$, *J. High Energy Phys.* **06** (2001) 050 [hep-lat/0103027].

[10] H.B. Meyer and M.J. Teper, *Glueball Regge trajectories and the pomeron: a lattice study*, hep-ph/0409183.

[11] P. de Forcrand, G. Schierholz, H. Schneider and M. Teper, *The string and its tension in* SU(3) *lattice gauge theory: towards definitive results*, *Phys. Lett.* **B 160** (1985) 137.

[12] M. Lüscher, K. Symanzik and P. Weisz, *Anomalies of the free loop wave equation in the WKB approximation*, *Nucl. Phys.* **B 173** (1980) 365.

[13] J.F. Arvis, *The exact q anti-q potential in Nambu string theory*, *Phys. Lett.* **B 127** (1983) 106.

[14] M. Lüscher and P. Weisz, *String excitation energies in* SU(N) *gauge theories beyond the free-string approximation*, *J. High Energy Phys.* **07** (2004) 014 [hep-th/0406205].

[15] O. Aharony, private communication.

[16] B. Lucini, M. Teper and U. Wenger, *The high temperature phase transition in* SU(N) *gauge theories*, *J. High Energy Phys.* **01** (2004) 061 [hep-lat/0307017].

[17] A.M. Polyakov, *Thermal properties of gauge fields and quark liberation*, *Phys. Lett.* **B 72** (1978) 477.

[18] M. Teper, *The finite temperature phase transition of* SU(2) *gauge fields in (2+1)-dimensions*, *Phys. Lett.* **B 313** (1993) 417 and unpublished.

[19] J. Engels et al., *A study of finite temperature gauge theory in (2+1) dimensions*, *Nucl. Phys.* **53** *(Proc. Suppl.)* (1997) 420 [hep-lat/9608099].

[20] M. Teper, unpublished.

[21] B. Lucini, M. Teper and U. Wenger, in preparation.

[22] H.B. Meyer, *Locality and statistical error reduction on correlation functions*, *J. High Energy Phys.* **01** (2003) 048 [hep-lat/0209145]; *The Yang-Mills spectrum from a 2-level algorithm*, *J. High Energy Phys.* **01** (2004) 030 [hep-lat/0312034].

[23] M. Lüscher and P. Weisz, *Locality and exponential error reduction in numerical lattice gauge theory*, *J. High Energy Phys.* **09** (2001) 010 [hep-lat/0108014].

[24] J. Ambjørn, P. Olesen and C. Peterson, *Stochastic confinement and dimensional reduction, 1. Four-dimensional* SU(2) *lattice gauge theory*, *Nucl. Phys.* **B 240** (1984) 189; *Stochastic confinement and dimensional reduction, 2. Three-dimensional* SU(2) *lattice gauge theory*, *Nucl. Phys.* **B 240** (1984) 533; *Three-dimensional lattice gauge theory and strings*, *Nucl. Phys.* **B 244** (1984) 262; *Observation of a string in three-dimensional* SU(2) *lattice gauge theory*, *Phys. Lett.* **B 142** (1984) 410.

[25] A. Hanany, M.J. Strassler and A. Zaffaroni, *Confinement and strings in mQCD*, *Nucl. Phys.* **B 513** (1998) 87 [hep-th/9707244];
M.J. Strassler, *QCD, supersymmetric QCD, lattice QCD and string theory: synthesis on the horizon?*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 120 [hep-lat/9810059]; *Messages for QCD from the superworld*, *Prog. Theor. Phys. Suppl.* **131** (1998) 439 [hep-lat/9803009].